Bayesian Methods Exercise III

Questions

1. Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

2. An arguement has broken out in the Senior Common Room over the probability that a fair coin, when tossed, will land on its edge. Denote this probability as θ. Professor Whitehead says that $\theta = 10^{-10}$, Professor Fearnhead says $\theta = 10^{-11}$, and Professor Tawn thinks $\theta = 10^{-12}$. You think that there is a probability of 3/8 that Professor Whitehead is right, you also think there is a probability of 3/8 that Professor Fearnhead is right, and a probability of 1/4 Professor Tawn is correct. To resolve the situation Professor Francis finds a coin and tosses it. Amazingly it lands on its edge. What is your posterior probability for each Professors value for θ being correct?

3. After the debacle described in Question 2 each professor quickly revises their estimates for θ. Now Professor Whitehead says that $\theta = 1/100$, Professor Fearnhead says $\theta = 1/500$, and Professor Tawn thinks $\theta = 1/1000$. Professor Francis pickes up his trusty coin and tosses it, on the 1096^{th} toss the coin lands on its edge. What are your posterior beliefs in each of the professors values for θ now?[†]

[†]It would be entirely possible to treat the posterior probabilities from the previous toss of the coin as priors in this instance. For the purposes of the question assume all previous results have been abandoned.

Solutions

1. Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

   Denote R as the event of "rain", and F as the event of "forecast of rain". From the question we can gather:

   (a) "*it has rained only 5 days each year*" which is the marginal probability of rain, or, $\Pr(R) = 5/365 = 1/73$.

   (b) "*When it actually rains, the weatherman correctly forecasts rain 90% of the time*" which is the conditional probability of forcast given rain, of $\Pr(F|R) = 9/10$.

   (c) "*When it doesn't rain, he incorrectly forecasts rain 10% of the time*", which is the probability of a forecast given no rain, or $\Pr(F|\bar{R}) = 1/10$

   It can either rain, or not rain, as $\Pr(R) + \Pr(\bar{R}) = 1$, we can infer that $\Pr(\bar{R}) = 72/73$.

   It can either rain, or not rain, during Marie's wedding. The forecast is for rain. So the probability we want is the probability of rain given the forecast of rain, or, $\Pr(R|F)$.

   We can write Bayes' theorum:

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B|A)}{\sum \Pr(A)\Pr(B|A)}$$

   Substituting into Bayes' theorum:

$$\begin{aligned}
\Pr(R|F) &= \frac{\Pr(R)\Pr(F|R)}{\Pr(R)\Pr(F|R) + \Pr(\bar{R})\Pr(F|\bar{R})} \\
&= \frac{1/73 \times 9/10}{(1/73 \times 9/10) + (72/73 \times 1/10)} \\
&= \frac{9/730}{9/730 + 72/730} \\
&= \frac{9}{9 + 72} \\
&= \frac{1}{9} \approx 0.111111 \approx 11\%
\end{aligned}$$

   Which is surprisingly low.

2. An arguement has broken out in the Senior Common Room over the probability that a fair coin, when tossed, will land on its edge. Denote this probability as θ. Professor Whitehead says that $\theta = 10^{-10}$, Professor Fearnhead says $\theta = 10^{-11}$, and Professor Tawn thinks $\theta = 10^{-10}$. You think that there is a probability of 3/8 that Professor Whitehead is right, you also think there is a probability of 3/8 that Professor Fearnhead is right, and a probability of 1/4 Professor Tawn is correct. To resolve the situation Professor Francis finds a coin and tosses it. Amazingly it lands on its edge. What is your posterior probability for each Professors value for θ being correct?

   Denote the event of a fair coin landing on its edge as E, let W denote Professor Whitehead, let F denote Professor Fearnhead and let T mean Professor Tawn.

   By the law of total probability:

$$\begin{aligned}\Pr(E) &= \Pr(W)\Pr(E|W) + \Pr(F)\Pr(E|F) + \Pr(T)\Pr(E|T) \\ &= 3/8 \times 10^{-10} + 3/8 \times 10^{-11} + 1/4 \times 10^{-12} \\ &= 4.15 \times 10^{-11}\end{aligned}$$

   So, your posterior probability for Professor Whiteheads value of θ being correct is:

$$\begin{aligned}\Pr(W|E) &= \frac{\Pr(W)\Pr(E|W)}{\Pr(E)} \\ &= \frac{3/8 \times 10^{-10}}{4.15 \times 10^{-11}} \\ &= \frac{3.75 \times 10^{-11}}{4.15 \times 10^{-11}} \\ &= 0.904\end{aligned}$$

   You think Professor Fearhhead's value of θ has a probability of:

$$\begin{aligned}\Pr(F|E) &= \frac{\Pr(F)\Pr(E|F)}{\Pr(E)} \\ &= \frac{3/8 \times 10^{-11}}{4.15 \times 10^{-11}} \\ &= \frac{3.75 \times 10^{-12}}{4.15 \times 10^{-11}} \\ &= 0.090\end{aligned}$$

   of being correct. And your posterior for Professor Tawn's value of θ being correct is:

$$
\begin{aligned}
\Pr(T|E) &= \frac{\Pr(T)\Pr(E|T)}{\Pr(E)} \\
&= \frac{1/4 \times 10^{-12}}{4.15 \times 10^{-11}} \\
&= \frac{2.5 \times 10^{-13}}{4.15 \times 10^{-11}} \\
&= 0.006
\end{aligned}
$$

So after the experiment you think there is over a 90% probability that Professor Whitehead's value of $\theta = 10^{-10}$ is correct. Adding all the posterior probabilities together we get $0.904 + 0.090 + 0.006 = 1$ which is what we would expect.

3. After the debacle described in Question 2 each professor quickly revises their estimates for θ. Now Professor Whitehead says that $\theta = 1/100$, Professor Fearnhead says $\theta = 1/500$, and Professor Tawn thinks $\theta = 1/1000$. Professor Francis pickes up his trusty coin and tosses it, on the 1096^{th} toss the coin lands on its edge. What are your posterior beliefs in each of the professors values for θ now?

   Denote the event of a fair coin landing on its edge exactly once from a sequence of 1096 tosses as E, let W denote Professor Whitehead, let F denote Professor Fearnhead and let T mean Professor Tawn.

   There have been 1096 tosses, exactly one of which has resulted in the coin landing upon its side. So for Professor Whitehead the whole sequence would happen with a probability:

$$\begin{aligned}
\Pr(E|W) &= (1-\theta)^{1095} \times \theta^1 \\
&= 1.661644 \times 10^{-05} \times 1/100 \\
&= 1.661644 \times 10^{-07}
\end{aligned}$$

   This is because there are 1095 events in which the coin did not land upon its edge. Each of these happened with probability $1 - \theta$. There was a single event whereby the coin landed on its edge, and this happened with probability θ. The likeihood of all these events, because they are independent, is the product.

   A similar calculation can be conducted for Professor Fearnhead

$$\begin{aligned}
\Pr(E|F) &= (1-\theta)^{1095} \times \theta^1 \\
&= 0.1116716 \times 1/500 \\
&= 0.0002233432
\end{aligned}$$

   and finally Professor Tawn:

$$\begin{aligned}
\Pr(E|F) &= (1-\theta)^{1095} \times \theta^1 \\
&= 0.3343564 \times 1/1000 \\
&= 0.0003343564
\end{aligned}$$

   Using the same arguement of total probability as used in Question 2:

$$\begin{aligned}
\Pr(E) &= \Pr(W)\Pr(E|W) + \Pr(F)\Pr(E|F) + \Pr(T)\Pr(E|F) \\
&= 3/8 \times 1.661644 \times 10^{-07} + 3/8 \times 0.0002233432 + 1/4 \times 0.0003343564 \\
&= 0.0001674051
\end{aligned}$$

   Then proceeding as in Question 2 for Professor Whitehead:

$$
\begin{aligned}
\Pr(W|E) &= \frac{\Pr(W)\Pr(E|W)}{\Pr(E)} \\
&= \frac{3/8 \times 1.661644 \times 10^{-7}}{0.0001674051} \\
&= \frac{6.231166 \times 10^{-8}}{0.0001674051} \\
&= 0.0003722208
\end{aligned}
$$

For Professor Fearnhead:

$$
\begin{aligned}
\Pr(W|E) &= \frac{\Pr(W)\Pr(E|W)}{\Pr(E)} \\
&= \frac{3/8 \times 0.0002233432}{0.0001674051} \\
&= \frac{8.37537e - 05 \times 10^{-5}}{0.0001674051} \\
&= 0.5003055
\end{aligned}
$$

And, finally for Professor Tawn:

$$
\begin{aligned}
\Pr(W|E) &= \frac{\Pr(W)\Pr(E|W)}{\Pr(E)} \\
&= \frac{1/4 \times 0.0003343564}{0.0001674051} \\
&= \frac{8.35891e \times 10^{-5}}{0.0001674051} \\
&= 0.4993223
\end{aligned}
$$

which checking by addition $0.0003722208 + 0.5003055 + 0.4993223 = 1$, and is expected.