Radical-theoretic approach to ring theory

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The Köthe nilradical

Throughout the rest of this talk $A$ stands for a fixed ring, unless otherwise stated.

Nilpotents are bad\(^1\): let us quotient them out from $A$:

**Definition (Köthe, 1930)**

*The Köthe radical (nilradical) $N(A)$ is the largest nil-ideal in $A$. i.e.*

\[ N(A) = \sum \{ I \vartriangleleft A : I \text{ is a nil ring} \}. \]

Some authors prefer the name *upper nilradical* for $N(A)$.

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\(^1\)are they?
The Köthe nilradical

One can describe $N(A)$ (using Zornification argument) by:

$$N(A) = \bigcap \{ I \triangleleft A : A/I \text{ is prime}^2 \text{ with no non-zero nil ideals} \}$$

Examples:

- $N(A[x_1, \ldots, x_n])$ consists of all polynomials with nilpotent coefficients.
- Given a *-ring $R$ such that $a^*a = 0$ implies that $a = 0$. Then $N(R) = 0$. In particular, $R$ can by any ring of operators on a Hilbert space closed under taking adjoints (self-adjoint).

Open problem (Köthe, 1930)

Does $N(A)$ contain all nil left-ideals in $A$?

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2 A ring $R$ is prime if every $a, b \in R$ and for each $r \in R$ $arb = 0$ implies that either $a = 0$ or $b = 0$. 

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The Baer radical

By virtue of the aforementioned description of $N(A)$ it is natural to define the **Baer radical**:

**Definition (Baer, 1943)**

$$\beta(A) = \bigcap \{l \triangleleft A : A/l \text{ is prime}\}.$$ 

Again, some authors prefer the name *lower radical* or *prime radical* for $\beta(A)$ and call rings with $\beta(A) = 0$ *semiprime*.

**Theorem**

- The ring $A$ is (semi-)prime if and only if $M_n(A)$ is (semi-)prime, $n > 0$.
- $\beta(M_n(A)) = M_n(\beta(A))$. 
The celebrated Jacobson radical

Define in $A$ a new operation $\circ$ by $a \circ b = a + b - ab$ and say that $I \triangleleft A$ is quasi-regular if $(I, \circ, 0)$ is a group.

Remark

An idempotent element $e$ of a ring is quasi-regular if and only if $e = 0$. Indeed, if $e = e^2$ and $x$ is its quasi-inverse, then $x + e - xe = 0$, thus $xe + e^2 - xe^2 = 0$. Therefore $e = e^2 = 0$.

The Jacobson radical $J(A)$ is defined as

Definition (Jacobson, 1945)

The (unique maximal) quasi-regular ideal

$$J(A) = \sum \{ I \triangleleft A : I \text{ is quasi-regular} \}.$$
Thus we have a simple but useful observation:

**Corollary**

A Jacobson radical ring \((A = J(A))\) does not contain non-zero idempotents.

We have the “intersection” characterisation of \(J(A)\): Recall that a ring \(R\) is *(left-)* primitive, if \(R\) contains a maximal left ideal \(I\) such that \(xR \subseteq I\) implies \(x = 0\). In this language

**Theorem**

\[
J(A) = \bigcap \{I \triangleleft A : A/I \text{ is primitive}\}
\]
The Brown–McCoy radical

Say that a ring $R$ is a $G$-ring if for each $a \in R$ we have

$$a \in \{ax - x + ya - y : x, y \in R\}.$$

**Definition (Brown–McCoy, 1947)**

The Brown-McCoy radical of $A$ is

$$G(A) = \sum \{ I \lhd A : I \text{ a G-ring} \}$$

$$= \bigcap \{ I \lhd A : A/I \text{ is a unital simple ring} \}$$
Are these notions really different?

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- Before we proceed to the relevant counterexamples let us summarize further similarities between these notions.
Subdirect decompositions

Given a family \( \{ R_t : t \in T \} \) of rings. Recall that the \textit{subdirect sum} \( \sum_{t \in T} \text{sub} R_t \) is the subring of the usual direct product \( \prod_{t \in T} R_t \) such that each induced projection \( p_t s \) is surjective, where

- \( p_t \) is the projection from \( \prod_{t \in T} R_t \) onto \( R_t \),
- \( s : \sum_{t \in T} R_t \to \prod_{t \in T} R_t \) is the inclusion mapping.

One should have in mind the next fact:

**Theorem**

A ring \( R \) is a subdirect sum of the family of rings \( \{ R_t : t \in T \} \) if and only if for each \( t \in T \) there exists ideal \( I_t \) such that \( R/I_t = R_t \) and \( \bigcap_{s \in S} I_s = 0 \).
We have

- $\beta(A) = 0 \iff A = \sum_{t \in T}^{\text{sub}} \{A/I: \text{each } A/I \text{ is a prime ring}\}$
- $N(A) = 0 \iff A = \sum_{t \in T}^{\text{sub}} \{A/I: \text{each } A/I \text{ is a prime ring with no non-zero nil ideals}\}$
- $J(A) = 0 \iff A = \sum_{t \in T}^{\text{sub}} \{A/I: \text{each } A/I \text{ is a primitive ring}\}$
- $G(A) = 0 \iff A = \sum_{t \in T}^{\text{sub}} \{A/I: \text{each } A/I \text{ is a simple unital ring}\}$
For a left-artinian ring $A$ (i.e. $A$ satisfies the descending chain condition on left-ideals) we have

$$\beta(A) = 0 \iff N(A) = 0 \iff J(A) = 0 \iff G(A) = 0$$

and the subdirect decomposition on a $A$ has finitely many summands

$$A = A_1 \boxplus A_2 \boxplus \ldots \boxplus A_n$$

where each $A_i$ is an artinian ring isomorphic to a matrix ring over a division ring.
A subring $R$ of the ring $\text{Hom}(V)$ (where $V$ is a vector space) is said to be *dense*\(^3\) in $\text{Hom}(V)$ if for each $n$-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ there exists a linear map $T \in R$ such that $Ta_i = b_i$.

**Theorem (Jacobson density theorem)**

A ring is primitive if and only if and only if it is isomorphic to a dense subring of $\text{Hom}(V)$, where $V$ is a vector space over a division ring.

One may think about the von Neumann bicommutant theorem as an operator-theoretic counterpart to this theorem.

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\(^3\)It should not be confused with the topological notion of density: finite rank operators on a separable Hilbert space $H$ are dense in this sense in $\text{Hom}(H)$ but are not topologically dense even in $B(H)$. 

Distinguishing the radicals

It is well known that if $V$ is a countably infinite dimensional vector space over a field, then the family $\mathcal{F}_V$ of finite rank operators on $V$ is a simple non-unital subring of $\text{Hom}(V)$. Using the Jacobson Density Theorem we see that:

$$0 = J(\mathcal{F}_V) \neq \mathcal{F}_V = G(\mathcal{F}_V)$$

Now consider the subring $A$ of $\mathbb{Q}$ given by

$$A = \left\{ \frac{2x}{2y + 1} : x, y \in \mathbb{Z}, \gcd(2x, 2y + 1) = 1 \right\}$$
For any \( a = \frac{2x}{2y+1} \in A \) the equation \( a \circ z = a + z - az = 0 \) has a solution

\[
z = \frac{a}{a - 1} = \frac{2x}{2(x - y - 1) + 1} \in A
\]

whence \( A \) is quasi regular. Note that \( A \) has no non-zero nilpotent elements, thus

\[
0 = N(A) \neq A = J(A).
\]

To show that \( \beta(A) \neq N(A) \) for some \( A \) it suffices to construct a nil ring which is prime as well. The construction was done by Baer and, independently, by Zelmanov and is beyond the scope of this talk. On the other hand, if \( A \) is noetherian, then \( \beta(A) = N(A) \).
Recall that $A$ is noetherian if it satisfies the ascending chain condition. In this case we have

**Theorem (Levitzki)**

*If $A$ is noetherian, then $N(A)$ is nilpotent*

Indeed, we note that the set of nil ideals of $A$ has a maximal element, say $N$. Since the sum of nilpotent ideals is nilpotent as well, $N$ contains every nilpotent ideal. Suppose that $N(A/N) \neq 0$. The ring $A/N$ is noetherian. Since each noetherian ring with a non-zero nil one-side ideal has a non-zero nilpotent ideal (this is highly non-trivial!) $A/N$ has a non-zero nilpotent ideal $I/N$, where $I$ is a suitable ideal of $A$. 
Distinguishing the radicals

In that case $I^k \subseteq N$ for some $k$, so $I$ is nil, hence $I = N$, but this is absurd as $I/N = 0$.

We have proved that $N(A) \subseteq N$. The opposite inclusion is clear, thus we have

**Corollary**

*If A is noetherian, then* $N(A) = \beta(A)$.

We have also

**Theorem**

*If A is commutative, then*

$$N(A) = \beta(A) = \sqrt{A} := \{ a \in A : a \text{ is nilpotent} \}.$$
In the unital case,
- \( J(A) = \bigcap\{ I \triangleleft A : I \text{ is maximal left-ideal} \} \)
- \( G(A) = \bigcap\{ I \triangleleft A : I \text{ is a maximal ideal} \} \)

For a Banach space \( E \), we have \( J(\mathcal{B}(E)) = 0 \), however the Brown-McCoy radical of \( \mathcal{B}(E) \) need not be trivial - take:
- \( E = \ell^2 \) (Calkin)
- \( E = c_0 \) or \( E = \ell^p, \ p \in [1, \infty) \) (Gohberg, Markus and Feldman)
- \( E = C[0, \omega^\omega] \) (folklore?)
- \( E = \) the James space (Laustsen)
- \( E = C[0, \omega_1] \) (K. and Laustsen)

For each \( E \) above, the space \( \mathcal{B}(E) \) has the unique maximal ideal.
Yood observed that the Jacobson radical of $\mathcal{B}(E)/\mathcal{F}(E)$ is non-zero for certain Banach spaces $E$. Kleinecke then defined the ideal of *inessential operators* by

$$\mathcal{E}(E) = \pi^{-1}J(\mathcal{B}(E)/\mathcal{F}(E)).$$

On the other hand, one can prove that

$$\pi^{-1}G(\mathcal{B}(E)/\mathcal{F}(E)) = G(\mathcal{B}(E))$$

for each infinite-dimensional Banach space $E$. 
Using the Gelfand-Naimark theorem one can prove that Jacobson radical of each $C^*$-algebra is trivial. Nevertheless, there are commutative radical Banach algebras (i.e. equal to its Jacobson radical). One can prove using the Spectral Radius Formula that a good example is $L^1[0,1]$ with the multiplication given by

$$(fg)(x) = \int_0^x f(x - y)g(y)dy.$$ 

Of course, a pathological example can be any Banach space $E$ with $x \cdot y = 0$ for each $x, y \in E$. The theory of radical Banach algebras is a huge branch of modern functional analysis.
References

1. R. Baer “Radical ring”, *Amer. J. Math.*, 65 (1943), 537–568


3. N. Jacobson ”The radical and semi-simplicity for arbitrary rings”, *Amer. J. Math.* 67 (1943), 300–320
